Mortgage loans: 1. Monthly payments & the total cost of borrowing

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Monthly payments

Let $u(t)$ be the amount owed at time t. During time δt the debt will have increased through the interest rate $r\,\delta t$ and decreased due to a payment $c\,\delta t$, so that

$$u(t+\delta t) = (1 + r\,\delta t)u(t) - c\,\delta t.$$

Dividing by δt and rearranging, one has

$$r\,u(t) - c = \frac{u(t+\delta t) - u(t)}{\delta t} \to u'(t).$$

The differential equation obtained by taking $\delta t \to 0$ has as solution $u(t) = [u(0) - c/r]\,e^{rt} + c/r$, where $u(0) = P - D$ (the amount initially owed is the amount borrowed: property price minus deposit). Since no more money is due at the end of the loan, $u(T) = 0$ and one obtains

$$c = \frac{r(P-D)}{1 - e^{-rT}}, \tag{1}$$

so that $u(t)$ can be written

$$u(t) = (P-D)\frac{e^{rT} - e^{rt}}{e^{rT} - 1}. \tag{2}$$

(Note that c and r are respectively payment per unit time and interest per unit time, so if one gives r in year^{-1}, e.g. $r = 5\%$ p.a., then c is the total reimbursement per year.)

For u to decrease with time, c must be greater than $r\,u(t)$ and in particular than its value at $t = 0$: $c > r(P - D)$. This is also clear from Eq. (1) since $1 - e^{-rT} < 1$. This sets a limit on how much can be borrowed ($P < D + c/r$); this is in a sense an 'interest-only' constraint as it does not require repayment of the principal.

Fraction of payments towards interest

We can remark that $r\,u(t)$ is the interest to be paid at t, so that $r\,u(t)/c$ represents the fraction of payments towards interest (the rest repays the principal):

$$\frac{r\,u(t)}{c} = 1 - e^{-r(T-t)}.$$

In particular the initial fraction of payments going towards equity is e^{-rT}; surprisingly, this is independent of the size of the deposit (although r, and perhaps T, will indirectly depend on it). If $T = 25$ years and $r = 5\%$ p.a. then about 28.7% of the initial payments are to pay principal (see Table I). The first payments go mostly towards interest and little capital is initially repaid.

A convenient approximation

Equation (2) can be rewritten

$$u(t) = (P-D)\frac{1 - e^{-r(T-t)}}{r(T-t)}\,\frac{rT}{1 - e^{-rT}}\,\frac{T-t}{T}.$$

The function $(1 - e^{-x})/x$ will reappear often. It can be approximated as an exponential by noting that

$$\begin{aligned}\ln\frac{1-e^{-x}}{x} &= \ln\left[1 - \frac{x}{2}\left(1 - \frac{x}{3} + \frac{x^2}{12} - \frac{x^3}{60}\right) + o(x^4)\right]\\ &= -\frac{x}{2}\left[1 - \frac{x}{12}\left(1 - \frac{x^2}{120}\right)\right] + o(x^4).\end{aligned}$$

(Note that the absence of a third order term and the smallness of the fourth order make the second order a very good approximation.) We thus have

$$\frac{1-e^{-x}}{x} \approx e^{-\alpha_x x} \tag{3}$$

with

$$\alpha_x = \frac{1}{2}\left(1 - \frac{x}{12}\right). \tag{4}$$

If $r = 5\%$ p.a. and $T = 25$ years then $\alpha_{rT} \approx 0.45$. The Taylor expansion of $(1 - e^{-x})/x$ is $1 - (1 - x/3 + x^2/12 - x^3/60)x/2 + o(x^4)$, whereas that of $\exp(-\alpha_x x)$ is $1 - (1 - x/3 + x^2/12 - x^3/57.6)x/2 + o(x^4)$: it is correct to third order and the fourth order is very close. The error is smaller than 5% for $x < 3.65$; if $x = rT$ this means $r < 14.5\%$ over 25 years (and if $r > 14.5\%$, you have bigger problems than the quality of the approximation).

Remarking that

$$\frac{1-e^{-x}}{x}\,\frac{y}{1-e^{-y}} \approx \exp\left[-\alpha_{x+y}(x-y)\right],$$

the repayments and the amount still owed are

$$c \approx (P-D)\frac{e^{\alpha_{rT} rT}}{T} \tag{5}$$

and

$$u(t) \approx (P-D)\frac{T-t}{T}\exp\!\left[\alpha_{r(2T-t)} rt\right]. \tag{6}$$

Table II shows that a spike in interest rate on a tracker mortgage increases monthly payments by about $(2\alpha_{rT} - 1/2)T\,\delta r\,c$. This means $0.4 \times 25 \times 2\% \times c = 0.2\,c$ for a two-percent increase (e.g. an increase of the Bank rate from 0.5% to 2.5%), i.e. an increase of repayments by about 20%. A decrease of the repayment period from 25 to 20 years increases repayments by only 10%.

T	r																
15 yrs	5.00%	5.42%	5.83%	6.25%	6.67%	7.08%	7.50%	7.92%	8.33%	8.75%	9.17%	9.58%	10.0%	10.4%	10.8%	11.3%	11.7%
20 yrs	3.75%	4.06%	4.38%	4.69%	5.00%	5.31%	5.63%	5.94%	6.25%	6.56%	6.88%	7.19%	7.50%	7.81%	8.13%	8.44%	8.75%
25 yrs	3.00%	3.25%	3.50%	3.75%	4.00%	4.25%	4.50%	4.75%	5.00%	5.25%	5.50%	5.75%	6.00%	6.25%	6.50%	6.75%	7.00%
30 yrs	2.50%	2.71%	2.92%	3.13%	3.33%	3.54%	3.75%	3.96%	4.17%	4.58%	4.58%	4.79%	5.00%	5.21%	5.42%	5.63%	5.83%
e^{rT}	2.12	2.25	2.40	2.55	2.72	2.89	3.08	3.28	3.49	3.72	3.96	4.21	4.48	4.77	5.08	5.41	5.75
e^{-rT}	0.472	0.444	0.417	0.392	0.368	0.346	0.325	0.305	0.287	0.269	0.253	0.238	0.223	0.210	0.197	0.185	0.174
$e^{\alpha_{rT} rT}$	1.42	1.46	1.50	1.54	1.58	1.62	1.67	1.71	1.75	1.80	1.84	1.89	1.93	1.98	2.02	2.07	2.12
$e^{-\alpha_{rT} rT}$	0.704	0.685	0.666	0.649	0.632	0.616	0.600	0.585	0.571	0.557	0.543	0.530	0.518	0.506	0.494	0.483	0.472
c	£ 474	£ 487	£ 500	£ 514	£ 525	£ 540	£ 555	£ 570	£ 585	£ 600	£ 615	£ 630	£ 645	£ 660	£ 675	£ 690	£ 705

TABLE I: Numerical values of e^{rT}, $e^{\alpha_{rT} rT}$ and c (in £/month) for $P - D =$ k£ 100 and $T = 25$ years for several combinations of r and T. All rates in the same column give the same value of rT.

$\frac{\partial \rightarrow}{\partial \downarrow}$	c	TCB
r	$\left(2\alpha_{rT} - \frac{1}{2}\right) c T \approx 10\, c$	$\left(2\alpha_{rT} - \frac{1}{2}\right) T \exp[\alpha_{(r+i)T}(r-i)T](P-D) \approx 15(P-D)$
T	$-\left(\frac{1}{2} + \frac{1}{rT} - 2\alpha_{rT}\right) c\, r \approx -2\%\, c$	$\left[2\alpha_{(r+i)T} - \frac{1}{2}\right](r-i) \exp[\alpha_{(r+i)T}(r-i)T](P-D) \approx 1.7\%(P-D)$
D	$-\dfrac{e^{\alpha_{rT} rT}}{T} \approx -7\%$	$-\left\{\exp[\alpha_{(r+i)T}(r-i)T] - 1\right\} \approx -0.40$
P	$\dfrac{e^{\alpha_{rT} rT}}{T} \approx 7\%$	$\exp[\alpha_{(r+i)T}(r-i)T] - 1 \approx 0.40$

TABLE II: Derivatives of the repayments and of the total cost of borrowing with respect to r, T, D and P. Numerical values for $r = 5\%$ p.a., $i = 2\%$ p.a., $T = 25$ years.

Total cost of borrowing

As a first approximation one can say that the total cost is the deposit D plus fees ϕ plus all the monthly payments minus the value of the property, i.e. $TCB = D + \phi + cT - P$. (Note that this is the cost of borrowing the money, i.e. it is the expense *on top* of what is paid for the property.) One should however remark that this implicitly assumes that the first payment of c and the last payment (20–30 years later) can be weighted equally. In fact one should discount later payments at some rate i (e.g. inflation), so that the total cost of borrowing is

$$TCB = D + \phi - P + \int_0^T c\, e^{-it} dt.$$

This gives

$$TCB = (P - D)\left(\frac{rT}{1 - e^{-rT}} \frac{1 - e^{-iT}}{iT} - 1\right) + \phi.$$

Using Eq. (3), the total cost is approximately

$$TCB \approx (P - D)\left[e^{\alpha_{(r+i)T}(r-i)T} - 1\right] + \phi. \qquad (7)$$

The rates r and i and the length of the loan T do not matter individually: only the product $(r - i)T$ does (at lowest order). (If $i = 3\%$ p.a. then an increase of r from 5% to 6% leads to an increase of $r - i$ from 2% to 3%, i.e. a nominal increase of 20% increases the total cost by 50%.) Similarly, P and D are not important: only the amount borrowed $(P - D)$ is.

Using that $P - D = P - K + \phi$ (deposit = capital − fees), Eq. (7) shows the true cost of fees:

$$TCB - TCB(\phi = 0) \approx \exp[\alpha_{(r+i)T}(r - i)T]\phi.$$

One can further approximate TCB :

$$TCB \approx \alpha_{2(2i-r)T}(r - i)T(P - D) + \phi. \qquad (8)$$

The effective (non-compounded) rate is $[1 + (r - 2i)T/6](r - i)/2$.

One should note that it was implicitly assumed that the value of the property would not change with time. In fact one should use its value at T in today's money: $Pe^{(i'-i)T}$. (Note that this does not take possible tax liabilities into account.) One can obtain a net total gain as the capital gain minus the cost of the loan: $NTG = P[e^{(i'-i)T} - 1] - TCB$, which gives

$$NTG \approx P\left[e^{(i'-i)T} - 1\right] - (P - D)\left[e^{\alpha_{(r+i)T}(r-i)T} - 1\right] - \phi.$$